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## **$B$ meson decay constant from quenched Lattice QCD**

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### Abstract

A lattice QCD calculation of the  $B$  meson decay constant is presented. In order to investigate the scaling violation associated with the heavy quark, parallel simulations are carried out employing both Wilson and the  $O(a)$ -improved clover actions for the heavy quark. The discretization errors due to the large  $b$  quark mass are estimated in a systematic way with the aid of the non-relativistic interpretation approach of El-Khadra, Kronfeld and Mackenzie. As our best value from the quenched simulations at  $\beta=5.9, 6.1$  and  $6.3$  we obtain  $f_B=163\pm 16$  MeV and  $f_{B_s}=175\pm 18$  MeV in the continuum limit where the error includes both statistical and systematic uncertainties.

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The  $B$  meson decay constant  $f_B$  is a fundamental quantity needed to extract the Cabibbo-Kobayashi-Maskawa matrix element  $V_{td}$  from experiments on  $B^0 - \bar{B}^0$  mixing. For this reason lattice QCD calculations of  $f_B$  have been pursued over several years, employing either relativistic [1,2] or non-relativistic [3] (including the static [4]) formulation for the  $b$  quark.

While there are a number of advantages with the relativistic formulation [1,2], its basic problem for calculations of  $f_B$  has lain in the difficulty of controlling systematic errors associated with heavy quark mass whose magnitude in lattice units exceeds unity for the  $b$  quark for a typical lattice spacing  $a^{-1} \approx 2 - 3$  GeV accessible in current simulations. The formalism proposed in Ref. [5], however, has shed a new light on this problem: these authors have pointed out that a Wilson-type lattice quark action for heavy quark can be reinterpreted as a non-relativistic Hamiltonian for an effective heavy quark field  $Q$  of the form,

$$H = \bar{Q} \left[ m_1 - \frac{\vec{D}^2}{2m_2} - \frac{i\vec{\sigma} \cdot \vec{B}}{2m_B} + O(1/m_Q^2) \right] Q. \quad (1)$$

where the effective heavy quark mass parameters  $m_i$  ( $i = 1, 2, B, \dots$ ) are functions of the bare quark mass  $m_Q$  and the coupling constant. In contrast to the continuum where these parameters are equal ( $m_1 = m_2 = m_B = \dots$ ), they mutually differ by  $O(am_Q)$  at finite lattice spacing, which represents  $O(am_Q)$  errors of the original action in the framework of the effective Hamiltonian (1). These parameters, however, are calculable in perturbation theory, and effects of  $O(am_Q)$  errors on  $f_B$  can be systematically analyzed. In particular, we observe that errors of  $O((m_2/m_B - 1)\Lambda_{QCD}/m_Q)$  for the Wilson action ( $m_B \neq m_2$ ) is reduced to  $O(\alpha_s \Lambda_{QCD}/m_Q, \Lambda_{QCD}^2/m_Q^2)$  for the  $O(a)$ -improved clover action [2], for which  $m_B = m_2$  holds at the tree level.

In this article we report on a calculation of the  $B$  meson decay constant in quenched lattice QCD with the relativistic formalism employing this “non-relativistic interpretation”. In order to study  $O(am_Q)$  systematic errors, we carry out a parallel set of simulations using both Wilson [6,7] and clover quark actions over a wide range of heavy quark mass and lattice spacing.

The parameters of our simulations are listed in Table I. The standard plaquette action is employed for gluons. For the clover coefficient we use the tadpole-modified [8] one-loop value [9]  $c_{sw} = 1/u_0^3[1 + 0.199\alpha_V(1/a)]$  where  $u_0 = P^{1/4}$  with  $P$  the average plaquette. The lattice size is chosen so that the physical size is approximately kept at  $L \approx 2$  fm. Seven values of the heavy quark hopping parameter are taken to cover the charm and bottom quark masses, and four values for light quark in a range  $0.4m_s - 1.4m_s$  with  $m_s$  the strange quark mass. The simulations have been carried out on the Fujitsu VPP500/80 supercomputer at KEK.

We extract the heavy-light decay constant  $f_P$  from the correlators of the axial vector current  $A_4$  and the pseudoscalar density  $P$  given by  $\langle A_4(t)P(0) \rangle$  and  $\langle P(t)P(0) \rangle$ . In order to reduce statistical errors of the correlators, which rapidly increase as  $am_Q$  increases, we employ the smeared pseudoscalar density  $P^S(x) = \sum_{\vec{r}} \phi(|\vec{r}|) \bar{Q}(x + \vec{r}) \gamma_5 q(x)$  on the gluon configurations fixed to the Coulomb gauge. The smearing function  $\phi(|\vec{r}|)$  is obtained by measuring the wave function of the pseudoscalar meson for each set of heavy and light quark masses. As a result we are able to isolate the ground state signal from a small time separation of  $t \approx 0.8$  fm.

We adopt for the heavy-light axial vector current  $\bar{q}\gamma_\mu\gamma_5 Q$  the one-loop renormalization factor  $Z_A(am_Q)$  newly calculated with full inclusion of the heavy quark mass dependence [10]. The calculation is available for both Wilson and clover actions, and it confirms the results of Refs. [11,12] made earlier for the Wilson action. We find that effects of finite  $am_Q$  are non-negligible: with  $Z_A(am_Q)$  evaluated with the coupling constant  $\alpha_V(1/a)$ ,  $f_B$  for the Wilson action is reduced by 5% (at  $\beta = 5.9$ ) to 2% (at  $\beta = 6.3$ ) compared to the value obtained with the  $Z_A$  factor with the mass dependence ignored, as employed in the previous studies. For the clover action the finite  $am_Q$  effect works in the opposite direction with a similar magnitude.

We remark that the field  $Q$  is related to the original field  $\Psi$  through

$$Q = e^{am_1/2}[1 + d_1\vec{\gamma} \cdot \vec{D}]\Psi, \quad (2)$$

where  $d_1$  is a known function of  $am_Q$  [5]. The KLM factor [5]  $e^{am_1/2}$  is evaluated including the  $m_Q$ -dependent one-loop correction [10]. We ignore the  $d_1\vec{\gamma} \cdot \vec{D}$  term, since its corrections to  $f_B$  is expected to be at most 1–2% due to a small value of  $d_1 \approx 0.1$ .

A non-trivial issue in lattice studies of heavy-light mesons is how to define their masses, since the pole mass directly measurable from meson propagators with zero spatial momentum suffers from large  $O(am_Q)$  errors. A possible choice is the kinetic mass defined by an expansion of the energy-momentum dispersion relation of the meson,

$$E_{\text{meson}}(\vec{p}) = m_{\text{pole}} + \frac{\vec{p}^2}{2m_{\text{kin}}} + O(\vec{p}^4). \quad (3)$$

The kinetic mass  $m_{\text{kin}}$ , however, receives corrections from  $O(\vec{p}^4)$  terms in (1) which are uncontrolled and hence suffer from a large  $O(am_Q)$  effect [13]. This leads to a pathology that the  $b$  quark mass cannot be determined consistently from heavy-light and heavy-heavy mesons [14,7].

An alternative choice may be to define a “kinetic mass” by taking the pole mass for a meson corrected by the difference of the kinetic and pole masses of the heavy quark  $m_2 - m_1$  [15,7],

$$m_{\text{kin}} \equiv m_{\text{pole}} + (m_2 - m_1) \quad (4)$$

This choice is motivated by the expectation that the binding energy of a heavy-light meson becomes independent of the heavy quark mass in the non-relativistic limit and  $(m_2 - m_1)$  should thus represent the difference between kinetic and pole masses of the meson. We find that the meson mass calculated in this way does not suffer from the pathology observed with  $m_{\text{kin}}$  defined by (3). We adopt this definition in our analyses using the one-loop perturbative result [10] for  $m_2 - m_1$ .

Let us now present our results. We plot  $\Phi(m_P) = (\alpha_s(m_P)/\alpha_s(m_B))^{2/\beta_0} f_P \sqrt{m_P}$  in Fig. 1 as a function of the inverse heavy-light meson mass  $m_P$  for both Wilson (open symbols) and clover (filled symbols) actions. The light quark mass is linearly extrapolated to the chiral limit, and  $\alpha_s(\mu)$  is calculated with the standard 2-loop definition where we employ the value  $\Lambda_{QCD} = 295$  MeV estimated from the  $\alpha_V$  coupling using the plaquette average [8].

There is an ambiguity in practice as to what mass scale is to be adopted to represent a quantity that has mass dimension. We prefer to use the scale that does not depend on the

quark sector to facilitate a direct comparison of the  $O(am_Q)$  errors with the two different quark actions on the common gauge configurations. Hence our natural choice is the string tension  $\sigma$ , and the ordinate is normalized by  $\sigma^{3/4}$  and the abscissa by  $\sigma^{1/2}$  in Fig. 1, where we employ the string tension of Ref. [16]. Vertical lines indicate the positions of the  $B$  and  $D$  mesons if one uses a phenomenological value  $\sqrt{\sigma} = 427$  MeV [17]. Data points plotted at  $1/m_P = 0$  are the static results [18], to which our data seem to converge towards the heavy quark mass limit. We observe that the Wilson results exhibit a small increase as the lattice spacing decreases, while the clover points at three values of  $\beta$  lie almost on a single curve.

An improved scaling behavior with the clover action is more clearly seen in Fig. 2, where we present the continuum extrapolation of  $f_B$  and  $f_D$ . Compared to scaling violation of 11-5% in our range of lattice spacing  $a^{-1} \approx 1.6\text{--}3$  GeV for the Wilson case, the clover data show a significantly reduced variation of 4-2% over the same range of lattice spacing. These magnitudes are common to  $f_B$  and  $f_D$ . The continuum values obtained by a linear extrapolation agree within the statistical error of about 5% between the two actions.

This agreement, however, does not necessarily mean that systematic  $O(am_Q)$  errors are all removed by the continuum extrapolation. Let us discuss this point for the Wilson action for the heavy quark. According to the non-relativistic Hamiltonian (1), the size of the leading  $O(am_Q)$  error in  $f_B$  is  $O((c_B - 1)\Lambda_{QCD}/m_Q)$  where  $c_B \equiv m_2/m_B$ . The tree level value  $c_B = 1/(1 + \sinh m_1 a)$  [5] is plotted in Fig. 3 as a function of  $m_2 a = e^{m_1 a} \sinh m_1 a / (1 + \sinh m_1 a)$ . For  $m_2 a \approx 2.9 - 1.5$ , corresponding to the  $b$  quark at  $\beta = 5.9 - 6.3$ ,  $|c_B - 1| \approx 0.7 - 0.5$ , and hence we expect an error of  $O(4 - 3\%)$  in  $f_B$  at our simulation points. If we linearly extrapolate  $c_B$  to the continuum limit  $m_2 a = 0$ ,  $|c_B - 1|$  decreases to 0.4, which implies an  $O(3\%)$  error left unremoved. For the  $D$  meson, the value of  $|c_B - 1|$  is smaller ( $|c_B - 1| \approx 0.4 - 0.3$  for the charm quark at  $m_2 a \approx 0.9 - 0.5$ ) and decreases faster, extrapolating to  $|c_B - 1| \approx 0.2$  at  $m_2 a = 0$ . Thus,  $O(am_Q)$  errors of  $O(7 - 5\%)$  for  $f_D$  at our simulation points reduces to  $O(3\%)$  in the continuum limit. This consideration indicates that the use of non-relativistic Hamiltonian inherently leaves a  $m_Q$ -dependent systematic error that cannot be removed by a linear extrapolation. We estimate that it is of the order of 3% for  $f_B$  and  $f_D$  in the continuum.

We need to consider two more sources of systematic errors, which can in principle be removed by the extrapolation procedure if the simulation is made at high precision but in practice are not removed from our results due to the insufficient statistics. One of them is  $m_Q$ -independent scaling violation, which is  $O(a\Lambda_{QCD})$  for the Wilson action. We take the value  $a\Lambda_{QCD} \approx 10\%$  at our smallest lattice spacing  $a^{-1} \approx 3$  GeV as an estimate of  $O(a\Lambda_{QCD})$  scaling violation effects. The other is the  $O(\alpha_V^2)$  uncertainty due to the use of one-loop value for  $Z_A$ , which is  $O(4\%)$  with  $\alpha_V(1/a) \approx 0.2$  at  $a^{-1} \approx 3$  GeV. Therefore, we expect a systematic error of the order of 10% in our results for the decay constant obtained with the continuum extrapolation.

This error analysis gives us some insight about the origin of the scaling violation observed in Fig. 2. We can conclude that the dominant part of the lattice spacing dependence comes from the  $m_Q$ -independent  $a\Lambda_{QCD}$  effect, since the  $m_Q$ -dependent errors  $O((c_B - 1)\Lambda_{QCD}/m_Q)$  diminishes only little towards the continuum limit and hence contributes little to the slope as a function of  $a$ . This leads us to expect that  $f_B$  and  $f_D$  exhibit a similar slope as a function of  $a$ , as we indeed observe in the figure. The size of scaling violation actually observed is within a factor of two from our estimate.

For the clover action the  $m_Q$ -dependent errors are reduced to  $O(\alpha_s \Lambda_{QCD}/m_Q)$  and  $O((\Lambda_{QCD}/m_Q)^2)$ . We estimate them to be  $O(1\%)$ . The scaling violation error is  $O(\alpha_s a \Lambda_{QCD})$  and  $O(a^2 \Lambda_{QCD}^2)$  which are of the order of 2%. Taking account of the  $O(\alpha_V^2)$  error from  $Z_A$  and that arising from the field rotation term  $d_1 \Lambda_{QCD}/m_Q \approx O(2\%)$  in (2), which is ignored in the present calculation, we expect systematic errors of order 5% for the decay constant from the clover action. The  $m_Q$ -independent scaling violation also dominates the  $a$  dependence of the decay constant, the contribution of  $m_Q$ -dependent errors being very small.

We now examine the question of how to set the physical scale of lattice spacing to calculate the decay constant. The most common in the literature is to use either  $\rho$  meson mass  $m_\rho$  or pion decay constant  $f_\pi$  to determine the lattice scale. In Fig. 4 we give the ratio of the lattice scale obtained with these quantities to that with the string tension. For the clover action the  $O(a)$ -improved axial vector current  $A_4 + c_A a \partial_4 P$  is used to measure  $f_\pi$  with the one-loop value for the coefficient  $c_A$  [9].

As expected, the slope of the ratio is much more gentle for the clover action compared to that for the Wilson action. The values in the continuum limit obtained by a linear extrapolation show a significant scatter, and the continuum limits of the ratio with the two actions disagree at the level of 5–10%. We ascribe this discrepancy mainly to smaller statistics of our Wilson simulation, and a resulting uncertainty in the continuum extrapolation.

The continuum value of the ratio need not be equal to unity in the quenched approximation; the disagreement may represent the systematic error due to quenching. Separating the quenching error from statistical and extrapolation uncertainties, however, is not possible with our present statistical accuracy. We then take the dispersion of the ratio in Fig. 4 as an uncertainty of the scale including the quenching error. We estimate it to be 10% for the Wilson action and 5% for the clover case.

We present our results for the physical value of the decay constant in Table II. Here we set the scale using the  $\rho$  meson mass. To obtain the ratio  $f_P/m_\rho$  in the continuum limit, we combine the continuum values of  $f_P/\sqrt{\sigma}$  and  $\sqrt{\sigma}/m_\rho$  obtained by a linear extrapolation as given in Figs. 2 and 4. A direct continuum extrapolation of  $f_P/m_\rho$  yields consistent results. The errors quoted in the parentheses are, in the order given, statistical, systematic and scale errors.

We take the result from the clover action to be our best estimate primarily because the uncertainty from scaling violation is smaller, but also because our statistical ensemble is larger for this case. Combining errors by quadrature we obtain  $f_B = 163 \pm 16$  MeV and  $f_{B_s} = 175 \pm 18$  MeV for the B meson decay constants. For the D meson we obtain  $f_D = 184 \pm 17$  MeV and  $f_{D_s} = 203 \pm 19$  MeV.

We have shown in this article that  $B$  meson decay constant within a 10% accuracy can be obtained with the  $O(a)$ -improved clover quark action in current lattice simulations at  $a^{-1} \approx 1.6 - 3$  GeV. The systematic error associated with the heavy quark is no longer the dominant source of uncertainty. The uncertainties in the lattice scale turns out to be more important in the present simulation. Time-consuming full QCD simulations are perhaps indispensable to go beyond the presently achieved accuracy in view of the fact that the scale uncertainty involving the quenching error will be the largest source of the uncertainty in the calculation of the heavy-light decay constant.

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# FIGURES

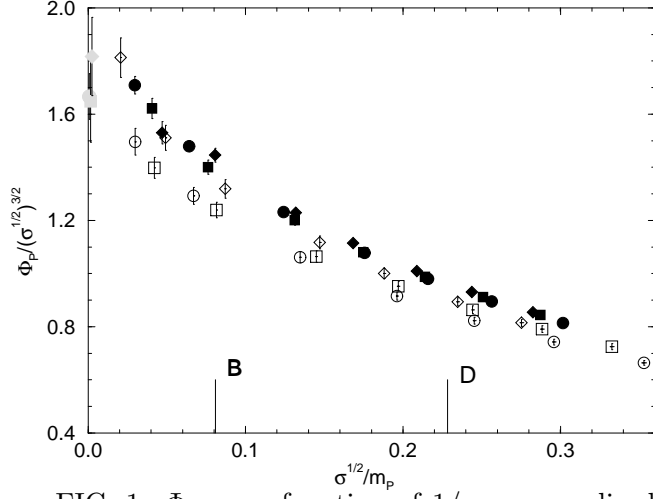


FIG. 1.  $\Phi_P$  as a function of  $1/m_P$  normalized by string tension  $\sigma$ . Filled symbols represent results with the clover action and open symbols with the Wilson action. Circles, squares and diamonds correspond to results at  $\beta=5.9, 6.1$  and  $6.3$ , respectively. Points at  $1/m_P = 0$  are static results [18] at the same set of  $\beta$ .

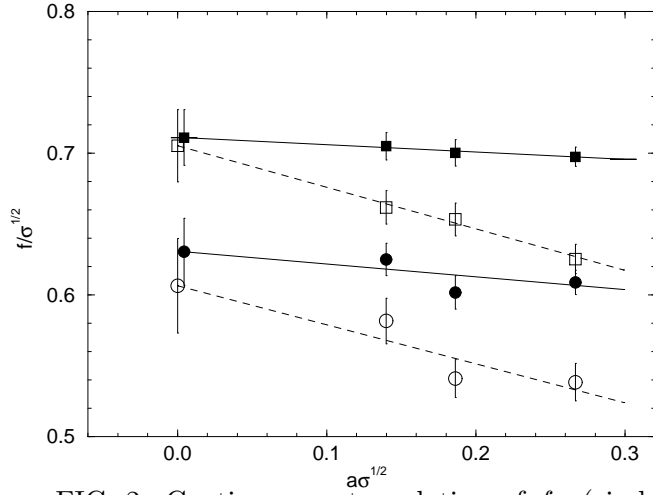


FIG. 2. Continuum extrapolation of  $f_B$  (circles) and  $f_D$  (squares). Filled symbols represent results with the clover action and open symbols with the Wilson action.



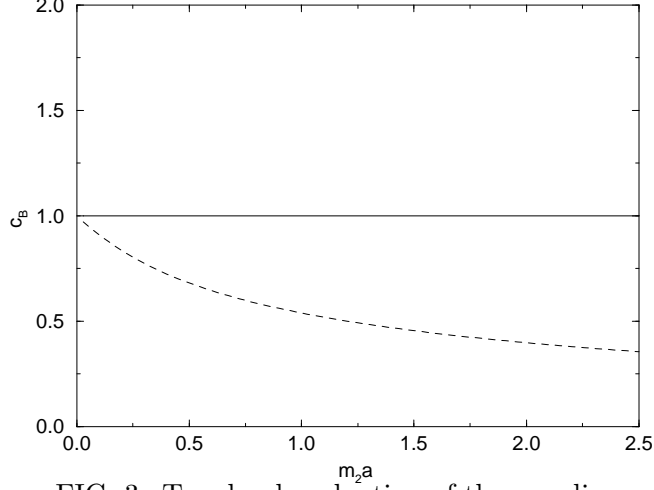


FIG. 3. Tree-level evaluation of the coupling  $c_B$  of the chromomagnetic interaction term in the non-relativistic effective Hamiltonian. Solid and dashed lines correspond to the clover and Wilson actions, respectively.

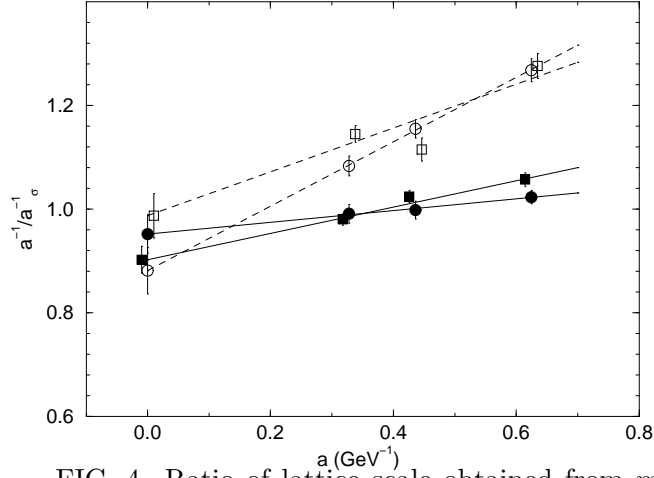


FIG. 4. Ratio of lattice scale obtained from  $m_\rho$  (circles) and from  $f_\pi$  (squares) to that from the string tension. Filled symbols represent results with the clover action and open symbols with the Wilson action.

# TABLES

TABLE I. Simulation parameters. The lattice scale quoted is estimated from  $m_\rho=770$  MeV.

action	$\beta$	5.9	6.1	6.3
	size	$16^3 \times 40$	$24^3 \times 64$	$32^3 \times 80$
Wilson	$N_{\text{conf}}$	150	100	100
	$1/a$ (GeV)	2.03(3)	2.65(4)	3.31(6)
clover	$N_{\text{conf}}$	540	200	166
	$c_{\text{sw}}$	1.580	1.525	1.484
	$1/a$ (GeV)	1.64(2)	2.29(4)	3.02(5)

TABLE II. Results for the decay constant in MeV unit.

	Wilson	clover
$f_B$	140(11)(15)(24)	163(9)(8)(11)
$f_{B_s}$	159(10)(17)(27)	175(9)(9)(13)
$f_D$	163(13)(18)(28)	184(9)(9)(12)
$f_{D_s}$	180(11)(20)(31)	203(9)(10)(14)